International Journal of Theoretical Physics, Vol. 21, Nos. 10/11, 1982

Generalizations of Gravitational Theory Based on Group Covariance[1]

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Received January 26, 1982

The mathematical structure, the field equations, and fundamentals of the kinematics of generalizations of general relativity based on semisimple invariance groups are presented. The structure is that of a generalized Kaluza–Klein theory with a subgroup as the gauge group. The group manifold with its Cartan–Killing metric forms the source-free solution. The gauge fields do not vanish even in this case and give rise to additional modes of free motion. The case of the de Sitter groups is presented as an example where the gauge field is tentatively assumed to mediate a spin interaction and give rise to spin motion. Generalization to the conformal group and a theory yielding features of Dirac's large-number hypothesis are discussed. The possibility of further generalizations to include fermions are pointed out. The Kaluza–Klein theory is formulated in terms of principal fibre bundles which need not to be trivial.

1. INTRODUCTION

The general theory of relativity indicates hardly any preference for a local invariance group. Globally there exist a multitude of Riemannian space-times which are realizations of various groups of transformations. Locally even a principle of equivalence can be formulated for most of their metrics in analogy to that of a locally Euclidean metric (Halpern, 1977). Spaces of constant curvature have the same number of Killing vectors as flat space, which is locally their limit of vanishing curvature, and they can be distinguished from it only if their curvature is large enough to be detectable.

The spinor equations of physics have been formulated for Minkowski space and the Poincaré group but a formulation for higher-dimensional homogeneous Lorentz groups has also been given (Dirac, 1935, 1936). One

[1]Presented at the Dirac Symposium, Loyola University, New Orleans, May 1981.

0020-7748/82/1000-0791$03.00/0

can thus say that the spinor formulation which was so successfully introduced to physics by Dirac does give preference to certain local invariance groups. Accepting this view, there remains still the choice between the Poincaré group, the simple de Sitter groups $SO(3,2)$ and $SO(4,1)$ and possibly some more complicated groups. Arguments for preference of a simple or at least semisimple invariance group can be based on speculations on the interrelation of all phenomena in nature which is not well compatible with Abelian invariant subgroups—and before all on mathematical beauty.

We assume in the present paper the existence of a semisimple local invariance group. Having made this crucial step, we will not assume or even expect to regain all of Poincaré group physics in the limit of a vanishing parameter. Too much of the fundamental mathematical structure is modified. Empirically the symmetry is broken and the structure of resulting gauge fields, for example, will not depend alone on the value of such parameters.

We go one step further by relating the physical quantities to projections on space-time of covariant quantities on the group manifold: Most realization spaces of Lie groups are the factor spaces of the group and a Lie subgroup (Steenrod, 1974). The manifolds of the de Sitter and anti-de Sitter universes, for example, are isomorphic to the factor spaces $SO(4,1)/SO(3,1)$ and $SO(3,2)/SO(3,1)$. A natural projection π exists from the points of the group manifold on the points of the factor space. The orbits of subgroups or even vector fields associated to the points of the group manifold (the latter are usually themselves factor spaces within different subgroups) can thus be projected on the points of the space-time manifold. The physical quantities depending on space-time we shall assume to be related to such projections. The next section presents the described features in detail with the help of principal fibre bundles. The metric of space-time of the universe is found as the projection of the Cartan–Killing metric of the group manifold. The trajectory of a test particle is approximated in this approach by the projection of a suitable geodesic of the group manifold (which is an orbit of a one-dimensional subgroup). This projection yields geodesic motion on space-time as a special case. The general motion has a helical component and is tentatively ascribed to a particle with spin. A classical free helical motion has of course not been observed in the laboratory; it may, however, exist in very large or very small domains. We speculate that quantization of the complicated nonlinear differential equations governing this motion (Halpern, 1978, 1979, 1980) may exclude other observability than that as the spin of elementary particles, similar as the particle wave character remains hidden to macroscopic observation. The anomalous motion results from the presence of "cosmological gauge fields" that do not vanish even with source-free solutions of the field equations. One could consider working with the simple equations of motion in which these fields and the "charges" occur explicitly. This would mean that one has to work with a "Vielbein" formalism supplementing the metric. The point of view adopted in the present work is that the equation should be formulated in such a way that the constants labeled by C^M do not occur in the four-dimensional formulation. Their elimination results in the mentioned more complicated equation. We also reject the alternative to make the parameters C^M a priori unobservably small. This would mean to cover up new features of the theory instead of exploring them. The C^M may have to occur in solutions of the equations but the hope is that the latter will show us how to quantize them. We have in general avoided the temptation to insert arbitrary constants into the theoretical structure. The radius of the universe in the $SO(3,2)$ theory has been gauged to unity and is assumed to be the reciprocal of the gravitational parameter.

A global formulation of Kaluza–Klein theories has been given somewhat prior independently by R. Kerner (1981). This paper and a preprint contain interesting field theoretic applications. Other applications to field theory were suggested by W. Mecklenburg (1979). None of the theories known to the author have hitherto attempted to consider the universal geometrical structure of an invariance group as a guide for the interactions which leads to the new features in the present paper. Some previous papers of the author have the same concept but still a mathematically incomplete formulation (Halpern, 1980b, 1981).

Section 4 shows how more general theories can be built on this mathematical structure. The presentation of D. Ebner (1981) made the author fully realize how only group theory need (and probably should) be used to extend the theory to include fermions.

A chief motive of the present research was the construction of a field theory in accord with Dirac's large-number hypothesis (Dirac, 1979). The latter requires not simply a variation of the gravitational parameter with time but rather a variation of all units determined by quantum physics relative to those of gravitational physics. A more general theory than that by Jordan and Thiry (1948) seems to be required to achieve this. The example of $SO(4,2)$ given in Section 4 shows how such theories based on the principles suggested here can be constructed. The investigation of the properties of such theories may in itself be stimulating. Testing their validity in nature can irrespective of the result only help to enrich our knowledge.

2. STRUCTURE OF THE THEORY ON THE GROUP MANIFOLD

We consider the manifold of an r-parameter semisimple Lie group G with $(r-k)$-parameter semisimple closed Lie subgroup H. The example we shall most frequently refer to is $G=SO(3,2)$ and $H=SO(3,1)$ with $k=4$.

The subgroup H implies the existence of a system of imprimitivity on G that means a set of $(r-k)$-dimensional subvarieties homeomorphic to H, one through each point of G, which are transformed into each other by translations of G (Eisenhart, 1933). Each subvariety is a coset of H and an orbit of H on G. Let G/H be the space of the left cosets of H in G and π the natural projection of G on G/H: $\pi b \to bH$ $b \in G$. They form the principal fibre bundle $P(G, \pi, G/H, H, H)$ with typical fibre H and group H, acting on H by left translation (Steenrod, 1974).

We can choose a base of left-invariant vector fields A_R $(R = 1 \cdots r)$ for G such that the last $r-k$ basis vectors A_M $(M > K)$ form a base of the subgroup H. We choose from now on capital or small Latin indices from $A \cdots K$ for values $1 \cdots K$, from $L \cdots Q$ for values $k+1 \cdots r$, and from $R \cdots Z$ for values $1 \cdots r$ ($r =$ dimension of G). The summation convention of double indices extends also over those values that are indicated by the choice of the indices. For example, $B_E B^E$ sums only over indices $E = 1 \cdots k$, $B_M B^M$ only over indices $M = k+1 \cdots r$, and $B_S B^S$ over indices $S = 1 \cdots r$. The basis vectors fulfill the commutation relations:

$$[A_R, A_S] = C^T_{RS} A_T \tag{1}$$

The dual base of left-invariant 1-forms A^R fulfill the Maurer–Cartan equations:

$$dA^R + \tfrac{1}{2} C^R_{ST} A^S \wedge A^T = 0 \tag{1a}$$

(d denotes the exterior derivative). A basis of right-invariant vector fields $\bar{A}_R$ and 1-forms $\bar{A}^R$ fulfill

$$[\bar{A}_R, \bar{A}_S] = C^T_{SR} \bar{A}^T \tag{1$'$}$$

$$d\bar{A}^R - \tfrac{1}{2} C^R_{ST} \bar{A}^S \wedge \bar{A}^T \tag{1a$'$}$$

A Riemannian metric γ is given on the manifold of the semisimple group, in our base its covariant components are

$$\gamma_{RS} = C^V_{RU} C^U_{SV} \tag{2}$$

(Cartan–Killing metric). The base can be chosen so that γ_{RS} assumes only values: $\gamma_{RS} = \pm \delta_{RS}$.

A metric g is then defined on the base manifold G/H of the principal fibre bundle P by the projection of the contravariant metric: $g = \pi' \gamma$. This projection is independent of the point on the fibre over the point x of the base manifold. This follows because all base vectors $A_{R'}\bar{A}_R$ are Killing vectors:

$$[\gamma, A_R] = [\gamma, \bar{A}_R] = 0 \tag{3}$$

This is in particular true for the vertical vectors A_M which lay in the fibre over x and have vanishing projections $\pi' A_M = 0$. A homeomorphic mapping φ_x: $P \to G/H \times H$ exists in a neighborhood $U_x(p)$ of any point p of the base manifold so that $\pi^{-1}U_x \to U_x \times H$ and canonical projection $\circ\varphi = \pi$. Local coordinates can therefore be introduced for which (Eisenhart, 1933)

$$A^i_M = 0, \qquad (\pi'\gamma)^{mt} = 0, \qquad (\pi'\gamma)^{ij} = g^{ij} = \gamma^{ij} \tag{4}$$

(consider designation of indices given before). Equations (2) and (3) lead then to

$$\gamma^{ij}_{,t} A^t_M = 0 \tag{4a}$$

(comma denotes ordinary derivative). The principal fibre bundle P is trivial in most of the examples dealt with, so that $P = G/H \times H$ is globally valid. The choice of a de Sitter group for G and of $SO(3,1)$ for H yields the corresponding de Sitter universe with its metric as the base space.

We are able to construct a 1-form ω_u on P with values in the Lie algebra of H:

$$\omega_u = e_M A^M(u), \qquad e_M = L^{-1'} A_M(u) \tag{5}$$

with $u \in G$ a point on P. This form is left invariant and fulfills

$$\omega_u(\mathfrak{v}) = L_u^{-1'} \mathfrak{v}_u \tag{5a}$$

if $q \in H$, ut remains on the same fibre as u, and we can identify in the present case

$$\tilde{R}_q u = uq \tag{6}$$

the right action of H on P (Nomizu, 1956[2]) because the bundle space is itself the group manifold of G.

[2]This old presentation seems still to be one of the best short readable accounts with least printing errors. For a new presentation see Y. Choquet, C. DeWitt, M. Dillard (1977). Analysis, Manifolds and Physics, (North Holland).

A left-invariant connection with horizontal vector space V_h with base $\{A_E\}$ $(E=1\cdots k)$ exists if the commutators of the A_M $(M>k)$ with every A_E lay in V_h (Nomizu, 1956).

The left invariance of the connection form ω [equation (5)] implies

$$\omega_{\tilde{R}_q u}\left(\tilde{R}'_q \mathfrak{v}_u\right) = L_{uq}^{-1\prime} R'_q \mathfrak{v}_u = L_q^{-1\prime} R'_q L_u^{-1\prime} \mathfrak{v}_u \tag{6a}$$

A (local) cross section $q(x), x\in B$, may be chosen. One finds then from equation (6a) by equating equal components of $\mathfrak{v}$ in a local natural base:

$$\omega\left(\tilde{R}_q u\right)_t = Ad(q^{-1})\omega(u)_t - A^M_m(q) q^m_{,t} \tag{6b}$$

q depends only on the first k coordinates which correspond to those of B so that the inhomogenous term on the right vanishes for $t>k$. A^M_m of this term depends only on q not on u.

The curvature 2-form Ω,

$$\Omega = d\omega + [\omega, \omega] \tag{7}$$

transforms homogeneously

$$\Omega\left(\tilde{R}_q u\right) = Ad(q^{-1})\Omega(u) \tag{7a}$$

Because of the Maurer–Cartan equations (1a), Ω does not vanish on the group manifold. The components for a natural base are

$$\Omega_{ik} = \tfrac{1}{2} C^M_{EF} A^E_i \wedge A^F_k e_M \qquad (E, F \leqslant K) \tag{8}$$

The principal bundle P can be identified with the bundle of Lorentz frames over B: The commutation relations equation (1) yield for the differential of the variation of the horizontal base vectors with the points on a fibre over B in a natural base:

$$\frac{\partial A^i_E}{\partial y^m} A^m_M = -C^F_{EM} A^i_F \tag{1b}$$

This corresponds to a Lorentz transformation of these vectors which are orthogonal with the Cartan–Killing metric γ. The same is true for their projection on the base B with the projected metric g.

The Lorentz bundle is a reduction of the frame bundle. Our connection on P results thus in a linear connection.

To see the relation let us consider the k-dimensional surface formed by the trajectories of all the horizontal left-invariant vectors through a point P_0

of P. As will be discussed later, each trajectory is a geodesic of γ in P and its projection on B is a geodesic of g. The corresponding linear connection can because of this differ from the Christoffel connection of g at most by a torsion tensor. We can see that the torsion must vanish at the origin (which can be chosen arbitrarily) if in the limit of infinitesimal displacement the projection of the left-invariant horizontal vectors are parallel displaced with the Christoffel connection of the corresponding projections at the origin. This can be seen to be the case for our example of the de Sitter groups. The connection and the linear connection here form an entity.

The Cartan–Killing metric γ on the group manifold of a semisimple group fulfills the relation

$$R_{uv} = \frac{1}{4}\gamma_{uv} \qquad \text{or} \qquad R_{uv} - \frac{1}{2}\gamma_{uv}R + \left(\frac{r}{8} - \frac{1}{4}\right)\gamma_{uv} = 0 \tag{9}$$

3. GENERALIZATION TO A THEORY OF KALUZA–KLEIN TYPE

Equations (9) have the form of sourceless Einstein equations with cosmological member for the metric (2) of the group manifold. The physical units of length on the base manifold determine the physical magnitude of this constant and the related cosmological member of the equations for the projected metric on the base manifold. Making the radius of the de Sitter universes the unit leaves its cosmological constant of conventional magnitude.

We want to generalize to the presence of inhomogenous sources and general metrics on the base manifold. We consider thus more general solutions with the right-hand member of equations (9), relaxing the geometrical conditions on the r-dimensional manifold P somewhat: P remains a principal fibre bundle with the same typical fibre and group H and a base manifold B of the same topology as G/H. The metric γ is still of the form of equation (2) with the same structure constants—however, of the base vectors B_R only the last $(r-k)$ B_M can be identified with the A_M. They are vertical vectors tangent to the fibre. Their commutation relations with all the other base vectors are still the same:

$$[B_M, B_R] = C^S_{MR}B_S \qquad (R, S = 1 \cdots r, M = k+1 \cdots r) \tag{1''}$$

but the commutation relations between the first k B_E are not prescribed. In a natural local base we have thus $A_M = B_M$ depending only on the last $(r-k)$ coordinates y^m. Equation (1b) remains valid; it is the only restriction on the first k components of B_E:

$$B^i_{E,m} B^m_M = -C^F_{EM} B^i_F \tag{1b'}$$

This allows a general metric

$$\gamma^{ik} = g^{ik} = B^i_R \gamma^{RS} B^k_S = B^i_E \gamma^{EF} B^k_F$$

on the base space. [Equations (4) and (4a) are valid in our coordinates.] For the dual base we have

$$B^M_n = A^M_n, \qquad B^E_n = 0, \qquad B^M_i \text{ arbitrary}$$

Equation (3) is now restricted to

$$[\gamma, B_M] = 0 \tag{3'}$$

These generalizations allow us still to define a connection with the Lie algebra valued form

$$\omega_u = e_M B^M(u), \qquad e_M = \tilde{L}^{-1}_q B(u)$$

if u is mapped into (x, q), $x \in B$, $q \in H$. The B_E form a base for the horizontal vector space V_h.

Because of equation (3′) the B_M remain Killing vectors and even equation (4a) is still valid, so that the metric projected onto B is well defined. The projection of the tangent vectors of a given geodesic in P on V_v is therefore at all its points equal (Halpern, 1981). The total horizontal component of the normalized tangent vector of a nonminimal geodesic is therefore also constant and a tangent vector horizontal at one point remains horizontal.

Let the tangent vector be

$$A = C^R B_R = \dot{y} \qquad (C^M = \text{const}) \tag{10}$$

$$\dot{A}^t = (C^R B^t_R)_{,z} B^z_S C^S = \ddot{y}^t \tag{10a}$$

$$\ddot{y}^t + \left\{ {t \atop vw} \right\} \dot{x}^v \dot{y}^w = \dot{C}^R B^t_R + (dB^W)^R_Q B^t_R C^Q C_W \tag{10b}$$

and for the projection on B with the same parameter

$$\ddot{x}^i + \left\{ {i \atop jk} \right\}^{(4)} \dot{x}^j \dot{x}^k = F^{Mi}{}_k \dot{x}^k C_M \tag{10c}$$

with

$$x = \pi y, \qquad C_M = \gamma_{MR} C^R, \qquad F^M = dB^M + \tfrac{1}{2} C^M_{PQ} B^P \wedge B^Q$$

$$(M, P, Q > K) \qquad (10\text{d})$$

The projection of a geodesic of P is thus a geodesic of B iff C^M vanishes. A term resembling the Lorentz force occurs if $C^M \neq 0$. According to equation (8), the corresponding analogs of the electromagnetic field F^M do not even vanish in case of the group manifold $P = G$. The generalized Kaluza–Klein theory considers the constants C^M as generalized charges (not, i.e., electric). The projection of a geodesics in P on B should be an approximation to a particle trajectory. [A physical object can in a dynamical theory hardly be well approximated by a point in R^r because the homogenuity condition equation (3′) of the metric is not well compatible with sources too restricted in the vertical direction.]

The theory admits thus an additional mode of free motion in the vacuum for particles with generalized charges. This motion for suitable values of the C^M related to the C^A will resemble the spiral motion of a charge in a magnetic field. This seems to be in contradiction to experience in empty space.

The author has repeatedly pointed out that this need not be so (Halpern, 1978, 1979, 1980). The charges are the quantized analog of C_M in equation (10c). Let us consider the de Sitter groups and tenatively associate the six C^M to the spin of a particle. The alternative motion must only manifest itself in microscopic domains somewhat in analogy to the wave character. The spin is particularly suited for such a model because it has a classical analog. The author has suggested to eliminate the C^M from equation (10c) and investigate the quantum analog of the resulting nonlinear system of differential equations of higher order.

We consider here only the kinematical aspects and leave the difficult problem of the spin motion as a conjecture. The geodesics are the orbits of one-dimensional subgroups on the group manifold. We may generalize the procedure and consider even the projection of the orbits of higher-dimensional subgroups or of their factor spaces from $P = G$ onto B. We obtain then the systems of partial differential equations of higher order which describe the projected surfaces on B.

The bundle P is also here still related to the Lorentz bundle: equation (1b′) shows that the components of the horizontal base vectors B_E at different points of the fibre over a point of B are related by a Lorentz transformation.

The canonical 1-form θ which maps vectors on P into the components of their projection in the frame determined by the point on the fiber can be expressed in a local natural frame:

$$\theta_r^i = \rho_j^i(p) A_r^E(p) A_E^j(p) = \rho_j^i(p)\delta_r^j \tag{11}$$

But the horizontal vectors A^t,

$$\rho_{j,t}^i A^t = 0 \tag{11a}$$

so that the torsion form

$$\Theta(\check{\mathfrak{u}}, \mathfrak{v}) = d\theta(\check{\mathfrak{u}}, \mathfrak{v}) + \omega(\check{\mathfrak{u}})\theta(\mathfrak{v}) - \omega(\mathfrak{v})\theta(\check{\mathfrak{u}}) = 0 \tag{12}$$

vanishes. The fact that horizontal geodesics are projected on geodesics on B suffices then to show that the linear connection associated to our connection obtained from the connection on P is the Christoffel connection. The connection and linear connection can also in this general case be regarded an entity.

We can choose for the Lagrangian of empty space the invariant density $\sqrt{g}R$ (R = curvature invariant in r dimensions). In a natural frame this quantity does not depend on the last $(r-h)$ coordinates because the metric fulfills the Killing equations (3′). We are thus able to use this Lagrangian even in a k-dimensional formulation and find apart from a cosmological member

$$\begin{aligned} \mathfrak{L} &= \sqrt{g}^{(k)}\left(R^{(k)} + \tfrac{1}{4}F^{ikM}F_{ikM}\right) \\ F_{ik}^M &= B_{k,i}^M - B_{i,k}^M + C_{PQ}^M B_i^P B_k^Q \end{aligned} \tag{13}$$

The de Sitter universes are solutions of the homogenous equations resulting in case of the de Sitter groups.

4. BRIEF DISCUSSION OF FURTHER GENERALIZATION AND RESULTS

The previous sections gave an account of the mathematical structure, the field equations, and some of the kinematics of a generalization of general relativity based on semisimple local invariance groups. The mathematical scheme is quite general but the example given was that of the de Sitter groups. The theory in four-dimensional form supplements the metric field by gauge fields related to it. Avoiding the introduction of arbitrary coupling constants, the coupling to the metric field is determined in our example only by the extention of the universe. The metric of the theory will thus in general modify solutions of Einstein's vacuum field equations by introducing a source. This may enable us to test the theory from precise observation of planetary motion alone. Crucial effects should be expected for spinning test bodies if the gauge field really mediates a spin interaction as its form would suggest.[3]

The gauge group of this model is the subgroup $SO(3,1)$ (Lorentz group). We arrive at a theory with the de Sitter group as gauge group if we choose for the group G a pseudo-orthogonal group in six dimensions—$SO(4,2)$, for example.

The base manifold B is then a five-dimensional space of constant curvature. The action of the conformal group on space-time can be obtained by choosing the null hyperquadrix where the constant R is zero (Dirac, 1936). This interesting case does not fit our construction in every respect, because the metric projected on the base in singular. We also want to get away from the Poincaré group and choose thus $R > 0$.

The metric obtained by projection on the five-dimensional base can be considered that of an (original) Kaluza–Klein theory if it has one suitable Killing vector field. This condition can be met by a further restriction on the metric of the bundle space. The five-dimensional formulation has already ten gauge fields corresponding to the six of our previous example. These fields are projected on the four-dimensional space-time (the base space of the five-dimensional bundle B which forms the base of the 15-dimensional bundle P). An additional gauge field results which is interpreted as the electromagnetic field. The latter need not be independent of all the other gauge fields (depending on the lift of the Killing vector field on B) thus allowing the formulation of more unified field theories. The five-dimensional formulation allows us to adopt straightforwardly the projective formalism of Veblen (1932) and the generalization to a theory with varying gravitational parameter of Jordan and Thiry (1948).[4] The performance of such a generalization poses no difficulty. One has, however, to keep in mind that a theory that does justice to Dirac's large-number hypothesis must not only exhibit a variation of the gravitational parameter relative to the electromagnetic but relative to all dimensionless parameters determined by

[3]An interaction with elementary particle spin alone should appear unlikely to a physicist because spin and orbital angular momentum can hardly be distinguished in complex systems. An apparent contradiction is eliminated by considering that for any orbital motion gravitational theory must take into account the field of the binding forces (e.g., gravitational field) which couples to the gauge field.

[4]The original idea was due to Einstein (unpublished). [See P. Bergmann, *Annals of Mathematics*, **49**, 255 (1948); see also the work of Brans and Dicke.]

quantum physics—thus also relative to angular momentum (Dirac, 1979). The present theory can meet these general requirements better than the original theory of Jordan because it describes other quantized magnitudes besides the charge. A change of these parameters relative to the gravitational requires a change of inclination of geodesics in the bundle space, in a timelike direction on the base space. (That means a change of the C^M of Section 3.) This change can be produced by a time dependance of the metric γ^{mn}.

The theory formulated until now does not describe the relation between spin and statistics. Supersymmetry can hardly be adopted to the present formalism which emerges from the properties of semisimple Lie groups. Dr. Ebner (1981) has recently shown how suitable Lie groups can extend and replace the supersymmetry transformations. New aspects for the generalization are thereby opened and work is in progress to explore them.

ACKNOWLEDGMENTS

I thank Professor W. T. van Est, Department of Mathematics, University of Amsterdam, for most valuable discussions and his help in establishing cross sections of principal fibre bundles. I am also indebted to Dr. S. Paneitz for valuable criticism of the original local formulation of the theory. To Dr. D. Ebner of the University of Konstanz, I am grateful for many valuable discussions and his enlightening group theoretical analysis of supersymmetries.

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